Example:

Determine the coefficients *h(n)* of a linear phase FIR filter of length *M=15* frequency response of which satisfies the condition:

$$H_r\left(\frac{2\pi k}{15}\right)=\begin{cases}1 & k=0,1,2,3\\ 0 & k=4,5,6,7\end{cases}$$

Solution:

$$H(n)=(-1)^n\, H_r\left(\frac{2\pi n}{M}\right)e^{j\frac{\pi n}{M}}=(-1)^n\, H_r\left(\frac{2\pi n}{15}\right)e^{j\frac{\pi n}{15}}$$ and taking into account $H(M-n)=\overline{H(n)}$ for $n=1,2,\ldots,M-1$

Step 1: *H(n)* computation for *n= 0, 1, 2, …, (M-1)/2* i.e. *n= 0, 1, 2, 3, 4, 5, 6, 7.*

$n=0$ $\quad H(0)=1$

$n=1$ $\quad H(1)=(-1)^1\, H_r\left(\frac{1.2\pi}{15}\right)e^{j\frac{1\pi}{15}}=-e^{j\frac{\pi}{15}}$ $\quad \phi_1=-\frac{14\pi}{15}$

$n=2$ $\quad H(2)=(-1)^2\, H_r\left(\frac{2.2\pi}{15}\right)e^{j\frac{2\pi}{15}}=e^{j\frac{2\pi}{15}}$ $\quad \phi_2=\frac{2\pi}{15}$

$n=3$ $\quad H(3)=(-1)^3\, H_r\left(\frac{3.2\pi}{15}\right)e^{j\frac{3\pi}{15}}=-e^{j\frac{3\pi}{15}}$ $\quad \phi_3=-\frac{12\pi}{15}$

$n=0,1,2,3$ $\quad |H(n)|=1$

$n=4,5,6,7$ $\quad |H(n)|=0$

Step 2: transfer function – basic expressions, *(M-1)/2=(15-1)/2=7*

$$H(z)=H_1(z)H_2(z)$$

$$H_1(z)=\frac{1-z^{-M}}{M} \qquad H_2(z)=\frac{H(0)}{1-z^{-1}}+\sum_{k=1}^{\frac{M-1}{2}}H_{2k}(z) \qquad H_{20}(z)=\frac{H(0)}{1-z^{-1}}$$

$$H_{2k}(z)=\frac{A(k)-B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}}$$

$$A(k)=H(k)+H(M-k)=H(k)+\overline{H(k)}=2\,\mathrm{Re}\left[H(h)\right]\in R$$

$$B(k)=H(k)e^{-j2\pi k/M}+H(M-k)e^{j2\pi k/M}=2\left|H(k)\right|\cos\left(\phi_k-2\pi k/M\right)\in R$$

Step 3: *A(k)* computation

$$A(1) = 2\,\mathrm{Re}\left[H(1)\right] = \text{-}1.9563$$
$$A(2) = 2\,\mathrm{Re}\left[H(2)\right] = 1.8271$$
$$A(3) = 2\,\mathrm{Re}\left[H(3)\right] = \text{-}1.6180$$

Step 4: *B(k)* computation

$$B(k) = 2\cos\left(\phi_k - 2\pi k/15\right)$$

$$B(1) = 2\cos\left(\phi_1 - 2\pi/15\right) = 2\cos\left(-14\pi/15 - 2\pi/15\right) = 2\cos\left(-16\pi/15\right) = \text{-}1.9563$$
$$B(2) = 2\cos\left(\phi_2 - 4\pi/15\right) = 2\cos\left(2\pi/15 - 4\pi/15\right) = 2\cos(-2\pi/15) = \ 1.8271$$
$$B(3) = 2\cos\left(\phi_3 - 6\pi/15\right) = 2\cos\left(-12\pi/15 - 6\pi/15\right) = 2\cos\left(-18\pi/15\right) = \text{-}1.6180$$

Step 5: $H_1(z)$ *and* $H_{20}(z)$ computation, *(M-1)/2=(15-1)/2=7*

$$H_1(z) = \frac{1 - z^{-15}}{15} \qquad H_{20}(z) = \frac{H(0)}{1 - z^{-1}}$$

Step 6: $H_{2k}(z)$ computation , *(M-1)/2=(15-1)/2=7,*

$\left|H(k)\right| = 0$ for $k = 4,5,6,7$ then $H_{2k}(z) = 0$ for $k = 4,5,6,7$

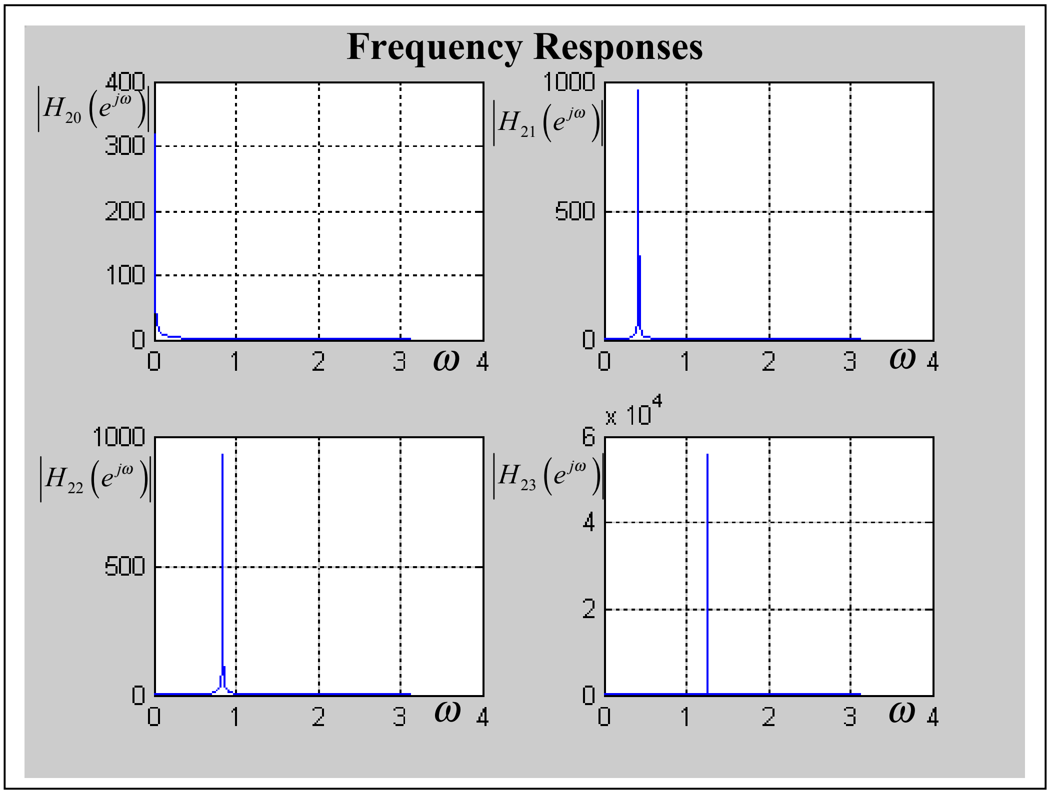
$$H_{2k}(z) = \frac{A(k) - B(k)z^{-1}}{1 - 2\cos(2\pi k/M)z^{-1} + z^{-2}}$$

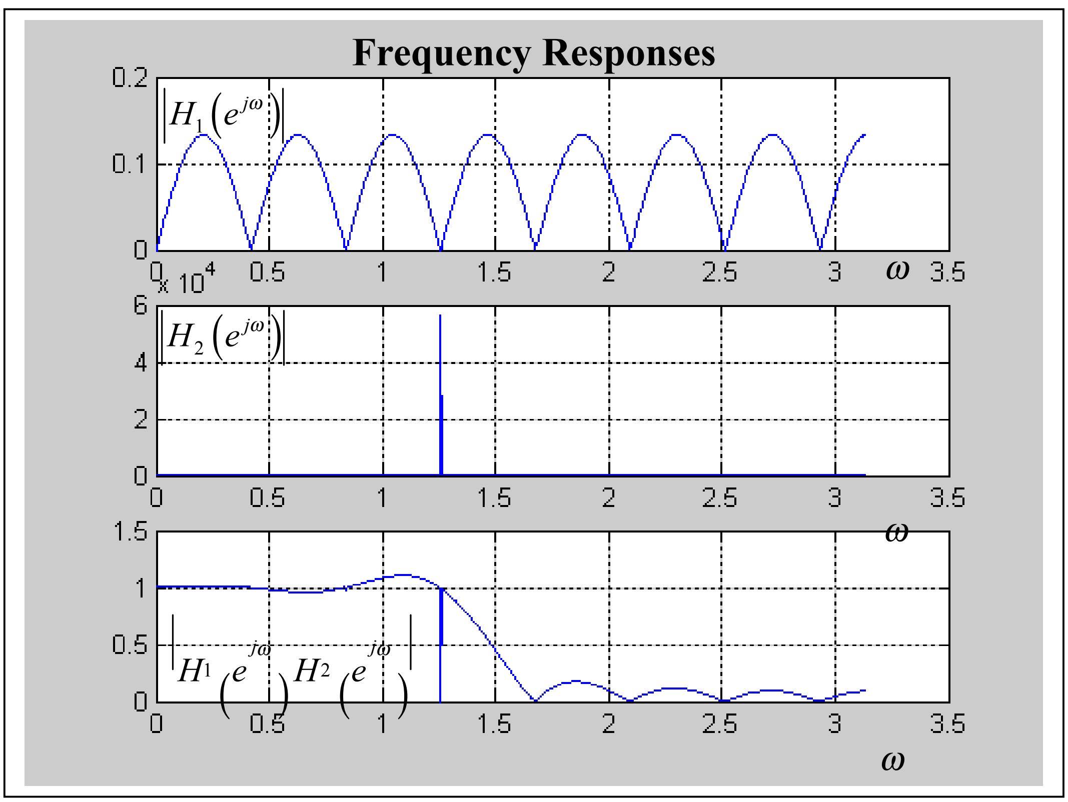
$$H_{21}(z) = \frac{\text{-}1.9563\text{+}1.9563z^{-1}}{1 - 1.8271z^{-1} + z^{-2}} \quad H_{22}(z) = \frac{1.8271\text{-}1.8271z^{-1}}{1 - 1.3383z^{-1} + z^{-2}} \quad H_{23}(z) = \frac{\text{-}1.6180\text{+-}1.6180z^{-1}}{1 - 0.6180z^{-1} + z^{-2}}$$

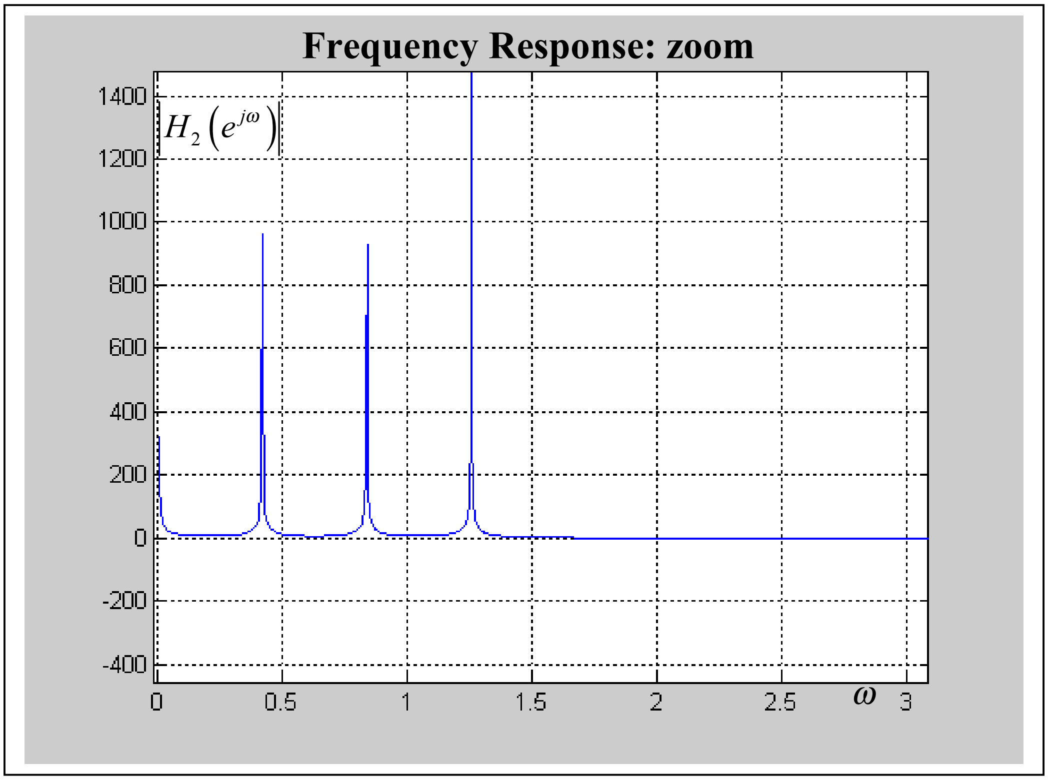
Step 7: result $H(z)$

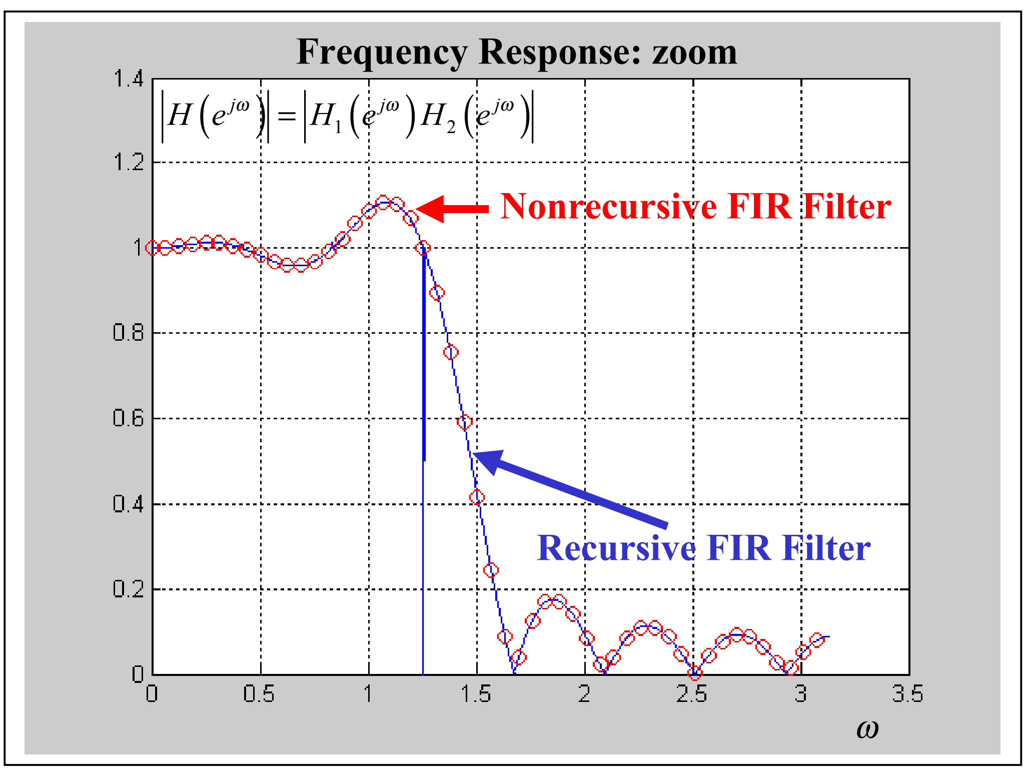
$$H(z) = \frac{1 - z^{-15}}{15}\left[\frac{1}{1 - z^{-1}} + \frac{\text{-}1.9563\text{+}1.9563z^{-1}}{1 - 1.8271z^{-1} + z^{-2}} + \frac{1.8271\text{-}1.8271z^{-1}}{1 - 1.3383z^{-1} + z^{-2}} + \frac{\text{-}1.6180\text{+}1.6180z^{-1}}{1 - 0.6180z^{-1} + z^{-2}}\right]$$

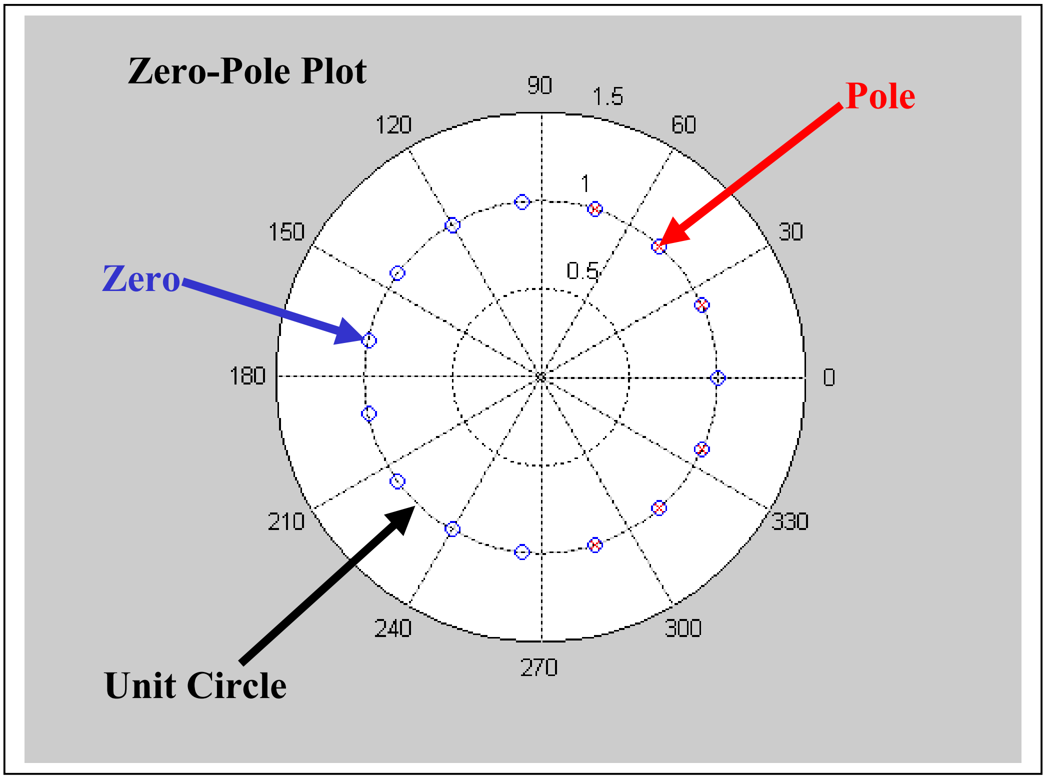
The different frequency responses and zero-pole plot are given in the next figures.


Frequency Responses
$|H_{20}(e^{j\omega})|$
$|H_{21}(e^{j\omega})|$
$|H_{22}(e^{j\omega})|$
$|H_{23}(e^{j\omega})|$
x 10^4
ω


Frequency Responses
$|H_1(e^{j\omega})|$
$|H_2(e^{j\omega})|$
$|H_1(e^{j\omega})H_2(e^{j\omega})|$
x 10^4
ω


Frequency Response: zoom
$|H_2(e^{j\omega})|$
1400
1200
1000
800
600
400
200
0
-200
-400
0
0.5
1
1.5
2
2.5
3
ω


Frequency Response: zoom
$|H(e^{j\omega})| = |H_1(e^{j\omega})H_2(e^{j\omega})|$
Nonrecursive FIR Filter
Recursive FIR Filter
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
ω



Zero-Pole Plot
90
1.5
120
60
Pole
1
150
30
Zero
0.5
180
0
210
330
240
300
270
Unit Circle