SECTION 3

The Fast Fourier Transform

3.1 Motivation

"Since there are two independent variables (time and frequency) in the Fourier transform, dividing (or decimating) the DFT into smaller ones can be done in two ways."

Upon closer examination of Eqn. 2-10, it becomes clear that for every frequency point, N-1 complex summations and N complex multiplications need to be evaluated. Since there are N frequency points to be evaluated, this gives a total of N(N-1) complex sums, and N^2 complex multiplications. Counting two real sums for every complex one, and four real multiplications plus two real summations for every complex multiplication, gives a total of $4N^2$- 2N real summations and 4 N^2 real multiplications.

The above numbers grow rapidly for increasing N. For N=1024 (1024-point DFT), 4,194,304 real multiplications are required. If this is computed on a DSP56001/ DSP56002 with a 27-MHz clock, it takes 0.31 seconds just to execute that many real multiplications. Since the DFT computation needs to be completed by the time the next 1024 data points are collected for real-time performance, the sampling rate is limited to a maximum of 3.3 kHz. Obviously, faster solutions are needed.

3.2 Divide and Conquer

A faster algorithm for computing the DFT can easily be derived. The principle behind this is very simple. As illustrated in Figure 3-1, a square of half the linear dimension of a larger square has one-fourth the surface area. This is because the surface area is proportional to the square of the linear dimensions of the square. Similarly, the number of multiplications needed to compute the DFT is proportional to the square of the DFT's length (N). Thus, if we could replace the DFT over N points by two DFTs over N/2 points, computations would be reduced in order of magnitude of 0.5 (=0.25+0.25).

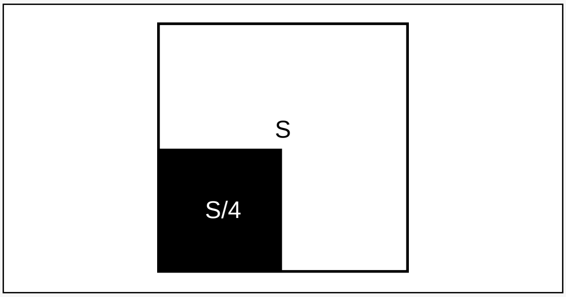

Figure 3-1 The FFT principle in layman's terms

Since there are two independent variables (time and frequency) in the Fourier transform, dividing (or decimating) the DFT into smaller ones can be done in two ways. We can attempt to represent an N-point transform in terms of DFTs over half the number (N/2) of time-samples. This approach is appropriately called the decimation-in-time or DIT approach. Alternatively, the N-point DFT can be represented in terms of DFTs with N/2 frequency samples. This approach is called the decimation-in-frequency or DIF approach.

3.3 The Decimation-in-Time and Decimation-in-Frequency Radix-2 Fast Fourier Transforms

It is easily shown that Eqn. 2-10 can be rewritten when N is even as:

$$X_N(k) = \sum_{r=0}^{(N/2)-1} \chi(2rT) e^{-j\frac{2\pi}{(N/2)}rk} + e^{-j\frac{2\pi}{N}} \sum_{r=0}^{(N/2)-1} X[(2r+1)T] e^{-j\frac{2\pi}{(N/2)}rk}$$

Eqn. 3-1

As illustrated in Figure 3-2, this expression shows how two N/2-point DFTs can be combined to obtain one N-point DFT. If N is an integer power of 2, this process can be repeated, as shown in Figure 3-3 and Figure 3-4, until a simple, two-point DFT is obtained. This gives rise to the flow diagram of a DIT fast Fourier transform (FFT) as shown in Figure 3-5, which represents a complete 8-point FFT computation.

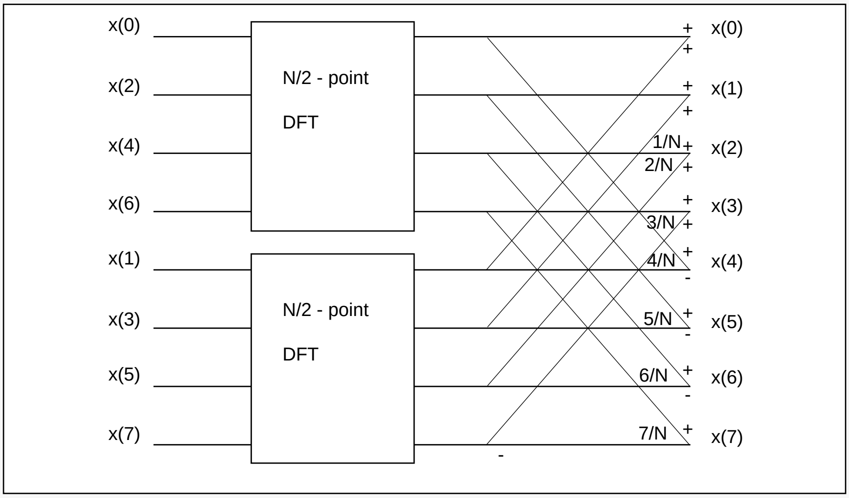

Figure 3-2 Decimation-in-Time of an N-Point

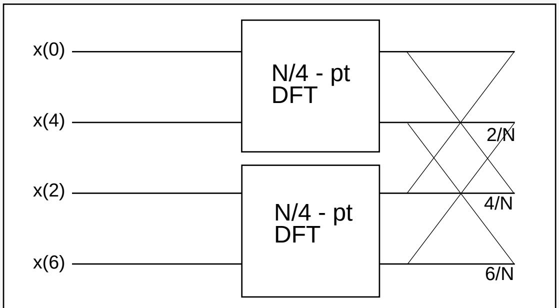

Figure 3-3 Decimation-in-Time FFT: Step Two

NOTE: k/N denotes multiplication by the “twiddle factors” $e^{-j\frac{2\pi}{N}k}$ throughout this document

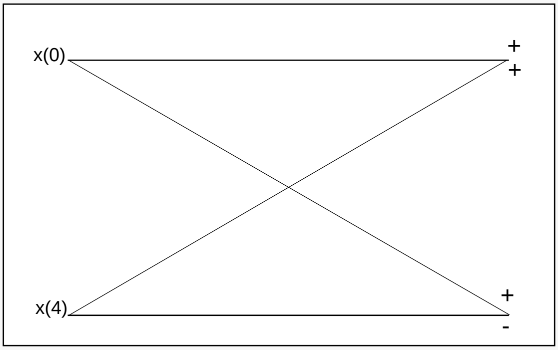

Figure 3-4 Decimation-in-Time FFT: Final Step (2-Point DFT)

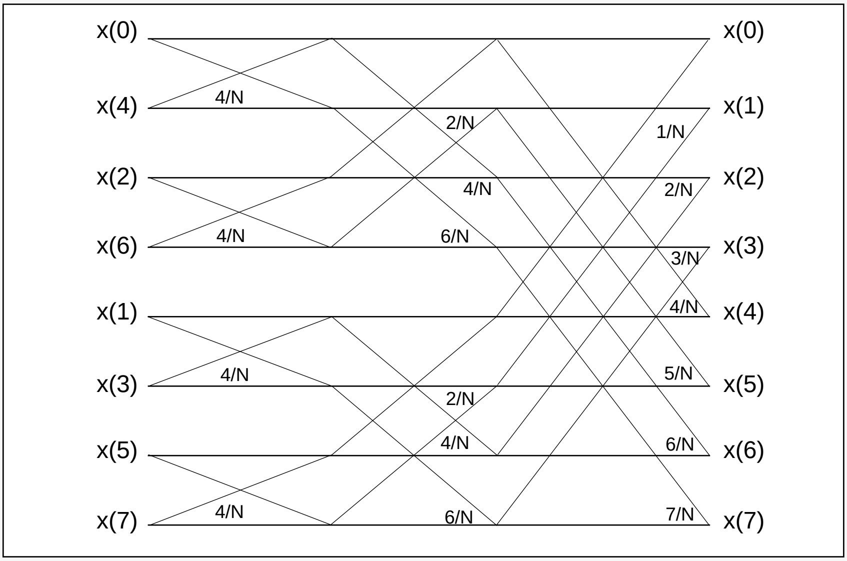

Figure 3-5 An 8-point, radix-2, Decimation-in-Time FFT

The basic flow diagram of Figure 3-5 can be further simplified by rearranging the terms in the basic building block (the butterfly) as in Figure 3-6. Also, it is seen from Figure 3-5 that input samples no longer occur in normal, sequential order. When the indices are represented in their binary equivalent, however, the input samples appear in “bit-reversed” order. Figure 3-8 shows how the diagram can be re-arranged for normally-ordered inputs and bit-reversed outputs.

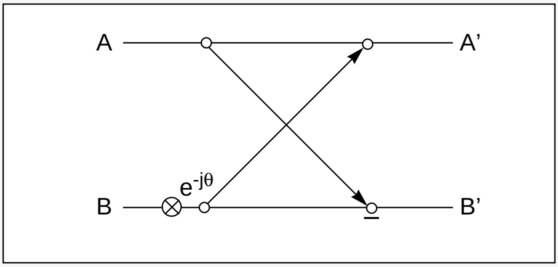

Figure 3-6 Rearrangement of the “butterfly” building block of the DIT FFT

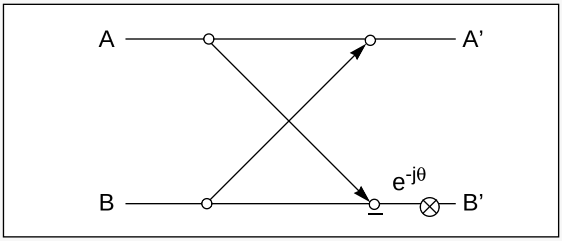

Figure 3-7 Rearrangement of the “butterfly” building block of the DIF FFT

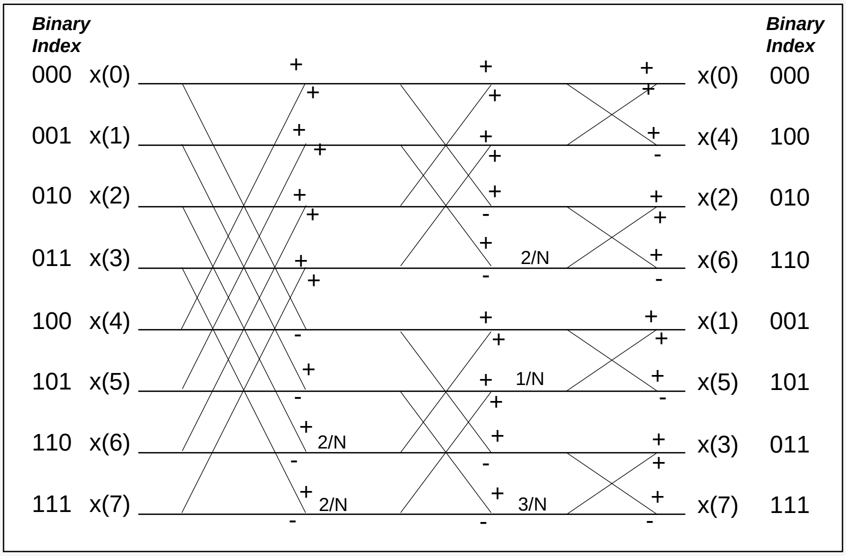

Figure 3-8 Rearrangement of the DIT computation of Figure 3-6

Figure 3-9 and Figure 3-10 show how the DFT with N frequency points can be obtained in terms of DFTs with a smaller number of frequency samples (decimation-in-frequency FFT). Note that the basic building block (butterfly) is different than for the DIT case (see Figure 3-10).

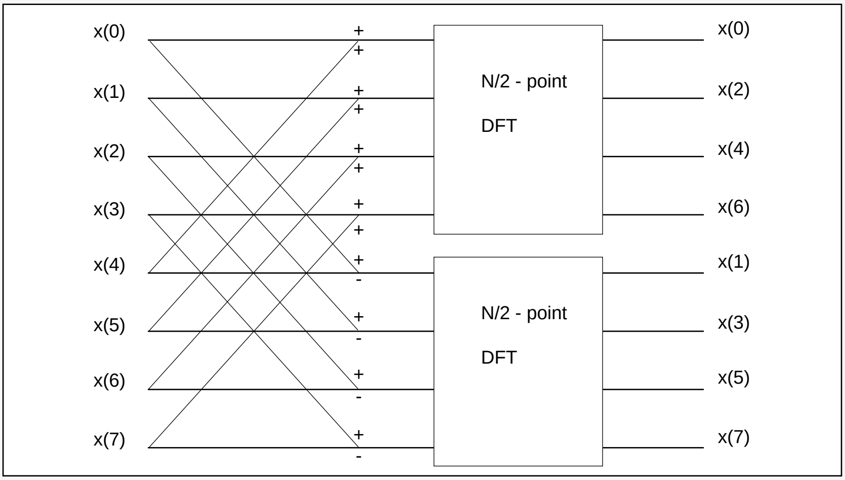

Figure 3-9 Decimation-in-Frequency concept

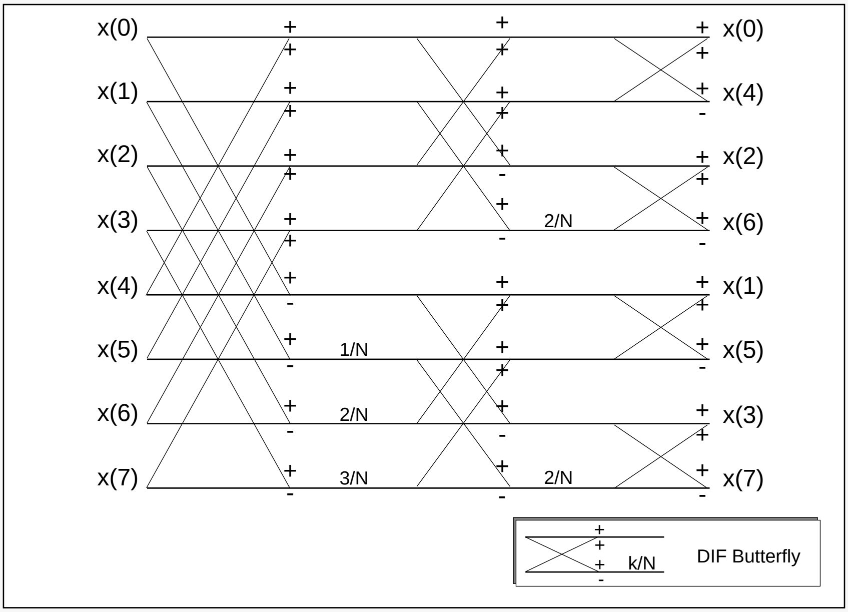

Figure 3-10 Complete 8-point radix-2 DIF FFT

3.4 The Decimation-in-Frequency Radix-2 Fast Fourier Transforms

If Eqn. 2-10 is decomposed from the frequency domain, we can show the following equations exist:

$$X_N(2k) = \sum_{r=0}^{(N/2)-1} [x(r) + x(r+N/2)]e^{-j\frac{2\pi rk}{N/2}}$$

Eqn. 3-2

$$X_N(2k+1) = \sum_{r=0}^{(N/2)-1} [x(r) - x(r+N/2)]e^{-j\frac{2\pi rk}{N/2}}e^{-j\frac{2\pi r}{N}}$$

Eqn. 3-3

The decimation in frequency butterfly is shown in Figure 3-9. ■